ELSEVIER

Journal of Pure and Applied Algebra 109 (1996) 1–22

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Duality theorems for finite semigroups

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Communicated by C.A. Weibel; received 29 April 1994; revised 9 February 1995

Abstract

If R is a ring graded by the semigroup S, then the smash product ring $R\#S^*$ can be constructed, and in many situations retains categorical "realization" properties analogous to those of the group-graded case. In particular, if S acts as endomorphisms on the ring A, then skew semigroup rings of the form $A * S^*$ and $S^* * A$ are graded by S, so we may form the "skew–smash" rings $(A * S^*) \# S^*$ and $(S^* * A) \# S^*$. On the other hand, S acts as endomorphisms on any ring of the form $R \# S^*$, so that the skew semigroup rings $(R \# S^*) * S^*$ and $S^* * (R \# S^*)$ may be produced. In particular we may perform this construction when R itself is a skew semigroup ring of the form $A * S^*$ or $S^* * A$, thereby yielding "skew–smash–skew" rings. In this article we analyze the resulting skew–smash and skew–smash–skew rings, and prove that each can be realized as a skew semigroup ring for an appropriate ring and (possibly new) semigroup. Inherent in our investigation is the description of a number of methods by which given semigroups can be used to produce new, related semigroups. As one consequence of our results we provide a broader context for some of the group-theoretic "duality" results of Cohen and Montgomery.

0. Introduction

During the 1980s many authors studied group-graded rings. Probably the most celebrated and useful results in the field are the two 'duality' theorems of Cohen and Montgomery [6, Theorems 3.2 and 3.5]. Historical precedents, along with a current renewed interest in graded rings, lead to an interest in some kind of duality theorems for semigroup-graded rings. The main aim of this article is the investigation of such results.

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[1] This article was written while the author was a member of G.N.S.A.G.A. of C.N.R. with partial financial support from M.U.R.S.T.

SSDI 0022-4049(96)00082-8

Indeed, the Cohen–Montgomery results may be viewed as results about semigroup-graded rings, as follows. For a finite group G let $S = \{e_{xy} \mid x, y \in G\} \cup \{z\}$; S has a semigroup structure defined by setting $e_{uv}e_{vw} = e_{uw}$ and setting all other products equal to z. Then the Cohen–Montgomery theorem for actions can be stated as follows: If G acts on the ring R as automorphisms then $R * G$ is G-graded, and the smash product $(R * G)\# G$ is isomorphic to the contracted semigroup ring RS^*. Similarly, their duality theorem for coactions takes the form: If R is a G-graded ring then G acts as automorphisms on the smash product $R\# G$, and $(R\# G) * G$ is isomorphic to the contracted semigroup ring RS^*. With this point of view in mind, we present in this article two types of results, of which we now give a brief synopsis.

Let R be a ring graded by the finite semigroup S. Analogous to the construction in the group-graded case (see e.g. [6]), we may form the smash product ring $R\# S^*$. In particular, if S is a semigroup which acts as ring endomorphisms on the ring A, then we may construct skew semigroup rings of the form $S * A$ and $A * S$ in a number of "natural" ways (we are purposely being rather imprecise with our notation here); regardless of the specific construction used, the resulting rings are always graded by S, so that we may form the "skew–smash" rings $(S * A)\# S^*$ and $(A * S)\# S^*$.

On the other hand, if R is a ring graded by the semigroup S, then there are natural actions of S as ring endomorphisms on $R\# S^*$. In particular, "smash–skew" rings of the form $S*(R\# S^*)$ and $(R\# S^*)*S$ may be produced. In the specific case in which R is a skew semigroup ring over S, the resulting rings can be viewed as "skew–smash–skew" rings.

The goal of this article is to describe explicitly skew–smash and skew–smash–skew rings. Throughout this article we will focus on two specific skew semigroup ring constructions; one quite general, the other arising in more structured contexts. For both of these constructions, we show (in Section 2) that the skew–smash rings $(S*A)\# S^*$ and $(A * S)\# S^*$ can be realized (respectively) as the rings $\widehat{S} * A$ and $A * \widehat{S}$ for a suitable semigroup $\widehat{S}$; see Theorems 2.3 and 2.5. Moreover, if S acts as automorphisms on the ring A then each of these rings is isomorphic to an incidence ring with coefficients in A. This last result may be viewed as a complete generalization to semigroups of the Cohen–Montgomery duality theorem for actions described above.

Unfortunately, there seems to be no 'nice' generalization to semigroups of the corresponding duality theorem for coactions. However, we are able to obtain concrete characterizations of certain types of rings in this setting. Specifically, in Section 3 we study the eight types of skew–smash–skew rings which arise in an examination of a ring A, semigroup S, and either one of the two specific types of skew semigroup ring constructions. We demonstrate that each of these eight rings can be realized as a skew semigroup ring over the ring A by utilizing a new semigroup which arises from the original semigroup S.

Additional information about semigroups can be found in [7]. Indeed, the results of this article may be viewed as a natural continuation of the discussion which appears in [7, Ch. 6]. Additional information about duality theorems for groups can be found in [8]. Loosely speaking, the approach we take here for semigroups mimics to a small degree the approach described for groups in [8, Section 1.2].

1. Preliminaries

Throughout this article S will denote a semigroup. If S contains a zero we will always denote it by z, and in this case we denote $S - \{z\}$ by S^*. For simplicity of exposition, *we will always assume that S is finite*; however, the reader will observe that a number of these results remain true more generally. The opposite semigroup of S will be denoted by S^{op}. Unless otherwise indicated, all functions and morphisms will be composed from left to right, so that $f \circ g$ (or simply fg) will mean "first f, then g". The letter A will denote an associative ring. We write $E(A)$ to denote the collection of ring endomorphisms of A (in case A is unital, we do *not* assume that such an endomorphism preserves the multiplicative identity of A), while $Aut(A)$ is used to denote the group of ring automorphisms of A.

The semigroup S is called *l.i.* (for "local identities") in case S^* contains a set of orthogonal idempotents E such that for each $g \in S^*$ there exist (necessarily unique) $e, e' \in E$ with $ege' = g$. In this case we sometimes denote e by e_g and e' by e'_g. We call S *right $*$-cancellative* in case for any three elements $f, g, h \in S^*$, if $fh = gh \in S^*$ then $f = g$. We call S a *category* in case S is l.i., and for any three elements $f, g, h \in S^*$, if $fg \in S^*$ and $gh \in S^*$ then $fgh \in S^*$.

If $\leq$ is a transitive relation on the set X, then the semigroup $X^{\leq}$ is defined to be the set of ordered pairs $\{(x, y) \in X \times X \mid x \leq y\} \cup \{z\}$, with multiplication given by setting $(x, y) \cdot (x', y') = (x, y')$ in case $y = x'$, z otherwise. The element (x, y) is sometimes denoted by $\leq_{x,y}$. If $(X, \leq)$ is a preorder, then $X^{\leq}$ is l.i. If A is any ring and $(X, \leq)$ is a preorder, then the incidence ring of X with coefficients in A is denoted $I(X, A)$. In particular, if $(X, \leq)$ is a totally ordered set with n elements, then $I(X, A)$ is the ring of $n \times n$ upper-triangular matrices over A, which we denote by $U_n(A)$.

If Γ is a directed graph then the *semigroup of Γ* is defined to be the set of directed paths (possibly of length zero) in Γ, together with z. Multiplication is given by juxtaposition when appropriate, z otherwise.

We say that a ring R is *graded* by S if there is a family $\{R_f \mid f \in S\}$ of additive subgroups of R such that $R = \bigoplus_{f \in S} R_f$, and for each pair f, g in S we have $R_f \cdot R_g \subseteq R_{fg}$. If R is graded by the l.i. semigroup S then we say R is *locally unital* in case for each $e \in E$ there exists $a_e \in R_e$ with the property that if $g \in S^*$ and $r \in R_g$ then $a_{e_g} r = r = r a_{e'_g}$. Any locally unital graded ring R is unital (since we have assumed that all semigroups are finite), with $1_R = \sum_{e \in E} a_e$.

When R is a ring graded by S then the *smash product ring $R \# S^*$* is defined to be the collection of S^*-square matrices of the form

$$\sum_{\substack{f, h \in S^* \\ fh \neq z}} r_f e_{fh,h}$$

under the usual matrix operations, where $r_f \in R_f$, and $r_f e_{fh,h}$ denotes the matrix which is r_f in the (fh,h) coordinate and 0 elsewhere. We point out that if S is l.i. and R is locally unital, then the ring $R\#S^*$ presented here is identical to the ring $R\Delta S^*$ given in [4]. Specifically, by [4, Corollary 2.27] in this situation we get that the category $R\#S^*$-mod is equivalent to the category R-gr_* (which consists of those S-graded left R-modules having zero z-component). Additionally, in this situation $R\#S^*$ is unital, with $1_{R\#S^*} = \sum_{l \in S^*} 1_{e_l} e_{l,l}$.

Definition 1.1. Let A be an associative ring, and let S be a finite semigroup. Suppose $\sigma : S^* \to E(A)$ (resp. $\sigma : S^* \to Aut(A)$) has the property that for any pair g, h in S with $gh \neq z$, $(gh)\sigma = (g)\sigma \circ (h)\sigma$. In this case we say that σ is *an action of* S^* *as endomorphisms* (resp. *automorphisms*) *on* A. For $a \in A$ and $h \in S$ we denote $(a)(h)\sigma$ by $a^{(h)\sigma}$.

We denote the abelian group $\bigoplus_{s \in S^*} A_s$ by $\langle S^* *_\sigma A\rangle$, where each $A_s = A$. For $s \in S$ and $a \in A$ we denote the element of $\langle S^* *_\sigma A\rangle$ which is a in the s-component and zero elsewhere by $s[a]$, or simply by sa. We define multiplication in $\langle S^* *_\sigma A\rangle$ by setting $ga \cdot hb = gh[a^{(h)\sigma}b]$ for each pair $g, h \in S$ having $gh \neq z$ and each pair $a, b \in A$, setting $ga \cdot hb = 0$ whenever $gh = z$, and extending linearly to all of $\langle S^* *_\sigma A\rangle$. $\langle S^* *_\sigma A\rangle$ is thus an associative ring.

If in addition A is unital, we denote the subgroup $\bigoplus_{s \in S^*} 1^{(s)\sigma} \cdot A_s$ of $\langle S^* *_\sigma A\rangle$ by $S^* *_\sigma A$. It is easy to show that $S^* *_\sigma A$ is a subring of $\langle S^* *_\sigma A\rangle$, and that these two rings are equal in case σ is an action as automorphisms on A.

We call any ring of the form $\langle S^* *_\sigma A\rangle$ or $S^* *_\sigma A$ a *skew semigroup ring* of S^* with coefficients in A. □

Definition 1.2. Let A be an associative ring, and let S be a finite semigroup. Suppose $\gamma : S^* \to E(A)$ (resp. $\gamma : S^* \to Aut(A)$) has the property that for any pair g, h in S with $gh \neq z$, $(gh)\gamma = (h)\gamma \circ (g)\gamma$. In this case we say that γ is *a reversing action of* S^* *as endomorphisms* (resp. *automorphisms*) *on* A. For $a \in A$ and $h \in S$ we denote $(a)(h)\gamma$ by $a^{(h)\gamma}$.

We denote the abelian group $\bigoplus_{s \in S^*} A_s$ by $\langle A *_\gamma S^*\rangle$, where each $A_s = A$. For $s \in S$ and $a \in A$ we denote the element of $\langle A *_\gamma S^*\rangle$ which is a in the s-component and zero elsewhere by $[a]s$, or simply by as. We define multiplication in $\langle A *_\gamma S^*\rangle$ by setting $ag \cdot bh = [ab^{(g)\gamma}]gh$ for each pair $g, h \in S$ having $gh \neq z$ and each pair $a, b \in A$, setting $ag \cdot bh = 0$ whenever $gh = z$, and extending linearly to all of $\langle A *_\gamma S^*\rangle$. $\langle A *_\gamma S^*\rangle$ is thus an associative ring.

If in addition A is unital, we denote the subgroup $\bigoplus_{s \in S^*} A_s \cdot 1^{(s)\gamma}$ of $\langle A *_\gamma S^*\rangle$ by $A *_\gamma S^*$. It is easy to show that $A *_\gamma S^*$ is a subring of $\langle A *_\gamma S^*\rangle$, and that these two rings are equal in case γ is a reversing action as automorphisms on A.

We call any ring of the form $\langle A *_\gamma S^*\rangle$ or $A *_\gamma S^*$ a *skew semigroup ring* of S^* with coefficients in A. □

Skew semigroup rings of the form $\langle S^* *_\sigma A\rangle$ or $\langle A *_\gamma S^*\rangle$ are the expected generalization of (contracted) semigroup rings. On the other hand, skew semigroup rings of the form $S^* *_\sigma A$ or $A *_\gamma S^*$ arise naturally in the study of endomorphism rings of graded modules. In addition, skew semigroup rings of this type are often unital, and possess some fairly interesting ring-theoretic properties. For additional information regarding skew semigroup rings of this type, see [2, 3].

Throughout this article we will be presenting isomorphisms between the aforementioned four types of skew semigroup rings and various other rings. In order to keep the notation and length of this article reasonably manageable, we will often spend most of our attention on a description of the isomorphisms for skew semigroup rings of the form $\langle S^* *_\sigma A\rangle$, and then let the reader supply the appropriate isomorphisms for rings of the form $S^* *_\sigma A$ (by restriction) or rings of the form $\langle A *_\gamma S^*\rangle$ or $A *_\gamma S^*$ (by symmetry).

Analogous to the situation for groups, skew semigroup rings of any of the four types are the prototypical examples of rings graded by the semigroup S, where for each $f \in S^*$ we define the f-graded component by setting $\langle S^* *_\sigma A\rangle_f = \{f[a] | a \in A\}$ and $\langle A *_\gamma S^*\rangle_f = \{[a]f | a \in A\}$. (We set the z-component of each of these rings equal to $\{0\}$.)

The supporting details for the following remarks can be found in [3, Section 2]. If S is an l.i. semigroup then any local unital S-graded ring is unital. Nonetheless, the rings $S^* *_\sigma A$ and $A *_\gamma S^*$ need not be unital or locally unital, even when S is l.i. However, if σ (resp. γ) has the additional property that $1^{(e)\sigma}a = a^{(e)\sigma}$ (resp. $a1^{(e)\gamma} = a^{(e)\gamma}$) for each $e \in E$ and $a \in A$, then $S^* *_\sigma A$ (resp. $A *_\gamma S^*$) is locally unital, hence unital. Under these hypotheses we say that σ (resp. γ) is a *locally unital action* (resp. *locally unital reversing action*). In particular, if S is an l.i. semigroup then any action or reversing action of S^* as automorphisms on a ring A is necessarily locally unital.

Example 1.3. Let k be a field, and let W denote the matrix ring

$$W = \begin{pmatrix} k & 0 & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

Let S denote the semigroup of the directed graph ${}_1\cdot \to \cdot_2$; so $S = \{1, \alpha, 2, z\}$ with 1 and 2 idempotent, $1\alpha = \alpha 2 = \alpha$, and all other products equal to z. Let $ae_{11}+be_{22}+ce_{23}+de_{33}$ denote an arbitrary element of W. We define $\sigma : S^* \to E(W)$ by setting:

$$(ae_{11} + be_{22} + ce_{23} + de_{33})^{(1)\sigma} = ae_{11}$$
$$(ae_{11} + be_{22} + ce_{23} + de_{33})^{(\alpha)\sigma} = ae_{22}$$
$$(ae_{11} + be_{22} + ce_{23} + de_{33})^{(2)\sigma} = be_{22} + ce_{23} + de_{33}.$$

A tedious check verifies that $(s)\sigma \in E(W)$ for each $s \in S^*$, and that σ is indeed an action of S^* as endomorphisms on W. Thus we may form the rings $\langle S^* *_\sigma W\rangle$ and $S^* *_\sigma W$.

By definition, each of the elements of $\langle S^* *_\sigma W\rangle$ can be written in the form $1w_1 + \alpha w_\alpha + 2w_2$ (with each $w_t \in W$). Furthermore, since $(1_W)^{(1)\sigma} = e_{11}$, $(1_W)^{(\alpha)\sigma} = e_{22}$, and

$(1_W)^{(2)\sigma} = e_{22} + e_{33}$, an easy computation verifies that the elements of $S^* *_\sigma W$ can be written in the form

$$1\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

where $a,b,c,d,e,f \in k$. Finally, another tedious computation verifies that $S^* *_\sigma W$ is in fact isomorphic to the ring $U_3(k)$ of upper triangular 3×3 matrices over k, by the map which takes

$$1\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}. \quad \square$$

If σ (resp. γ) is the function which associates the identity automorphism on the ring A with each element of S^*, then the resulting semigroup rings $\langle S^* *_\sigma A\rangle$ and $S^* *_\sigma A$ (resp. $\langle A *_\gamma S^*\rangle$ and $A *_\gamma S^*$) are equal; we denote this (contracted) semigroup ring simply by S^*A (resp. AS^*). In case $S = X^{\leq}$ for some preordered set X, then $S^*A \cong AS^* \cong I(X,A)$.

If σ is an action of S^* as endomorphisms on A, then σ may be viewed as a reversing action of $(S^{\mathrm{op}})^*$ on A in the obvious way. We denote the resulting skew semigroup ring by $A *_\sigma S^{o*}$. A similar statement holds for reversing actions. If S is a group then the rings $S^* *_\sigma A$ and $A *_\sigma S^{o*}$ are isomorphic; for general semigroups, however, they need not be (see for instance the remark following [3, Example 1.4]).

The following definitions are developed more fully in [3].

Definition 1.4. Let S be a semigroup, and let R be an S-graded ring.

(1) Suppose S is right $*$-cancellative. We define an action ρ of S^* as endomorphisms on $R\#S^*$ as follows: for each $h \in S^*$, $(h)\rho \in E(R\#S^*)$ is the linear extension of the function

$$(h)\rho: \quad r_f e_{fg,g} \longmapsto \begin{cases} r_f e_{fgh,gh} & \text{if } fgh \in S^*; \\ 0 & \text{otherwise.} \end{cases}$$

(The right $*$-cancellativity of S is used to show that each $(h)\rho$ is a ring homomorphism of $R\#S^*$.) Thus we may form the rings $\langle S^* *_\rho [R\#S^*]\rangle$ and $\langle [R\#S^*] *_\rho S^{o*}\rangle$. If $R\#S^*$ is unital (e.g. if S is l.i. and R is a locally unital) we may form the rings $S^* *_\rho [R\#S^*]$ and $[R\#S^*] *_\rho S^{o*}$ as well.

(2) We define a reversing action λ of S^* as endomorphisms on $R\#S^*$ as follows: for each $h \in S^*$, $(h)\lambda \in E(R\#S^*)$ is the linear extension of the function

$$(h)\lambda: \quad r_f e_{fg,g} \longmapsto \sum_{\substack{k \in S^* \\ kh=g}} r_f e_{fk,k}$$

(where we interpret the empty sum as 0). Thus we may form the rings $\langle [R\#S^*] *_\lambda S^* \rangle$ and $\langle S^{o*} *_\lambda [R\#S^*] \rangle$. If $R\#S^*$ is unital (e.g. if S is l.i. and R is locally unital) we may form the rings $[R\#S^*] *_\lambda S^*$ and $S^{o*} *_\lambda [R\#S^*]$ as well. □

When S is l.i. and R is locally unital the two actions and two reversing actions described in Definition 1.4 are locally unital, so that the four rings $S^* *_\rho [R\#S^*]$, $[R\#S^*] *_\rho S^{o*}$, $[R\#S^*] *_\lambda S^*$, and $S^{o*} *_\lambda [R\#S^*]$ are in fact unital.

2. "Skew–smash" constructions

If S is a semigroup for which there is an action (resp. reversing action) as endomorphisms on the ring A, then we may form the skew semigroup ring $\langle S^* * A \rangle$ (resp. $\langle A * S^* \rangle$) as described in the previous section. Any ring of this type is graded by S, so that we may in turn construct a ring of the form $\langle S^* * A \rangle \# S^*$ (resp. $\langle A * S^* \rangle \# S^*$). Similar statements hold for rings of the form $S^* * A$ (resp. $A * S^*$). Our goal in this section will be to concretely describe these resulting "skew–smash" rings. Specifically, we will show that each of these may be realized as a skew semigroup ring for a new semigroup $\widehat{S}$ with coefficients in A.

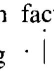
Definition 2.1. Let S be any semigroup. We form a new semigroup $\widehat{S}$ by setting $\widehat{S} = \{(s,x) \in S^* \times S^* \mid sx \in S^*\} \cup \{\widehat{z}\}$, and defining multiplication in $\widehat{S}$ by setting

$$\text{in } \widehat{S}: \quad (s,x)\cdot(s',x') = \begin{cases} (ss',x') & \text{if } x = s'x'\,; \\ \widehat{z} & \text{otherwise.} \end{cases}$$

It is easy to check that this indeed yields a semigroup structure on $\widehat{S}$. Moreover, if S is l.i., then so is $\widehat{S}$, with $\widehat{E} = \{(e_x, x) \mid x \in S^*\}$.

Now suppose $\sigma : S^* \to E(A)$ is an action of S^* as endomorphisms on the ring A. We define a new function $\widehat{\sigma} : \widehat{S}^* \to E(A)$ by setting $((s,x))\widehat{\sigma} = (s)\sigma$. It is straightforward to show that $\widehat{\sigma}$ is an action of $\widehat{S}^*$ as endomorphisms on A (and that $\widehat{\sigma}$ is locally unital in case S is l.i. and σ is locally unital). We therefore may in general form the skew semigroup ring $\langle \widehat{S}^* *_{\widehat{\sigma}} A \rangle$, and the ring $\widehat{S}^* *_{\widehat{\sigma}} A$ when appropriate.

Similarly, if $\gamma : S^* \to E(A)$ is a reversing action of S^* as endomorphisms on the ring A, we define a new function $\widehat{\gamma} : \widehat{S}^* \to E(A)$ by setting $((s,x))\widehat{\gamma} = (s)\gamma$. Then $\widehat{\gamma}$ is a reversing action of $\widehat{S}^*$ as endomorphisms on A (and $\widehat{\gamma}$ is locally unital in case S is l.i. and γ is locally unital). We therefore may in general form the skew semigroup ring $\langle A *_{\widehat{\gamma}} \widehat{S}^* \rangle$, and the ring $A *_{\widehat{\gamma}} \widehat{S}^*$ when appropriate. □

Example 2.2. Let S, W, and σ be the semigroup, ring, and action described in Example 1.3. It is easy to check that $\widehat{S} = \{(1,1),(1,\alpha),(\alpha,2),(2,2)\} \cup \{\widehat{z}\}$. In fact, $\widehat{S}$ is isomorphic to the semigroup arising from the partially ordered set having  as its Hasse diagram.

The elements of $\langle \widehat{S}^* *_{\widehat{\sigma}} W\rangle$ are of the form $(1,1)w_{11}+(1,\alpha)w_{1\alpha}+(\alpha,2)w_{\alpha 2}+(2,2)w_{22}$ (where each $w_{ab} \in W$), while $\widehat{S}^* *_{\widehat{\sigma}} W$ consists of elements of the form

$$(1,1)\begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (1,\alpha)\begin{pmatrix} k_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (\alpha,2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & k_3 & k_4 \\ 0 & 0 & 0 \end{pmatrix} + (2,2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & k_5 & k_6 \\ 0 & 0 & k_7 \end{pmatrix}$$

(where $k_i \in k$ for $1 \le i \le 7$). We will describe $\widehat{S}^* *_{\widehat{\sigma}} W$ as a ring of matrices (indeed, as a smash product ring) in Example 2.4 below. □

We offer the remarks in this paragraph as a sidelight observation regarding one of the distinctions between a semigroup T and its nonzero elements T^*. Using the notation of the previous example, we set $T = \widehat{S}$; then $\widehat{\sigma}$ is an action of T^* as endomorphisms on W. We claim that $\widehat{\sigma}$ cannot be extended to a semigroup homomorphism from T to $E(W)$. To see this, just suppose $\widehat{\sigma}$ could be extended. Since $(1,1)(\alpha,2) = \widehat{z}$ in T we would necessarily have

$$(\widehat{z})\widehat{\sigma} = ((1,1)\cdot(\alpha,2))\widehat{\sigma} = ((1,1))\widehat{\sigma} \circ ((\alpha,2))\widehat{\sigma} = (1)\sigma \circ (\alpha)\sigma = (\alpha)\sigma,$$

which is *not* the zero endomorphism on W. On the other hand, $(1,1)\cdot(2,2) = \widehat{z}$ in T, so that

$$(\widehat{z})\widehat{\sigma} = ((1,1)\cdot(2,2))\widehat{\sigma} = ((1,1))\widehat{\sigma} \circ ((2,2))\widehat{\sigma} = (1)\sigma \circ (2)\sigma,$$

which *is* the zero endomorphism on W.

The following theorem is the first main result of this article. In it, we show that there is a strong connection between the skew semigroup ring $\widehat{S}^* *_{\widehat{\sigma}} A$ and the star-smash process.

Theorem 2.3. *Let S be a semigroup, and let σ be an action of S^* as endomorphisms on the ring A. Then, with $\widehat{\sigma}$ as described in Definition 2.1, there is an isomorphism of rings*

$$\langle \widehat{S}^* *_{\widehat{\sigma}} A\rangle \cong \langle S^* *_\sigma A\rangle \# S^*.$$

If in addition A is unital then this isomorphism restricts to an isomorphism of rings $\widehat{S}^* *_{\widehat{\sigma}} A \cong (S^* *_\sigma A) \# S^*$. Moreover, if S is l.i. and σ is locally unital, then each of these latter two rings is unital, and the isomorphism is as unital rings.

Proof. We define $\Theta : \langle \widehat{S}^* *_{\widehat{\sigma}} A\rangle \to \langle S^* *_\sigma A\rangle \# S^*$ by setting $((s,x)a)\,\Theta = sae_{sx,x}$ for each $(s,x) \in \widehat{S}^*$ and each $a \in A$, and extending linearly. Then for any $(s,x),(s',x') \in \widehat{S}^*$ and $a,a' \in A$ we have

$$((s,x)a\cdot(s',x')a')\Theta$$
$$= \begin{cases} ((ss',x')a^{((s',x'))\widehat{\sigma}}a')\Theta & \text{if } x = s'x' \\ 0 & \text{if } x \ne s'x' \end{cases}$$

$$= \begin{cases} ss'a^{(s')\sigma}a'e_{ss'x',x'} & \text{if } x = s'x' \\ 0 & \text{if } x \neq s'x' \end{cases} \qquad \text{(using the definitions of } \widehat{\sigma} \text{ and } \Theta)$$

$$= \begin{cases} (sa \cdot s'a')e_{ss'x',x'} & \text{if } x = s'x' \\ 0 & \text{if } x \neq s'x' \end{cases} \qquad \text{(as the product is in } S^* *_\sigma A)$$

$$= sae_{sx,x} \cdot s'a'e_{s'x',x'}$$

$$= ((s,x)a)\Theta \cdot ((s',x')a')\Theta.$$

Thus Θ is a homomorphism. It is easy to see from the definitions of the appropriate rings that Θ is bijective.

Also, it is clear from the definitions that Θ restricts to an isomorphism from $\widehat{S}^* *_{\widehat{\sigma}} A$ to $(S^* *_\sigma A) \# S^*$. The final statement is straightforward. □

Example 2.4. With the notation as in the previous examples, the ring $\langle S^* *_\sigma W\rangle \# S^*$ is, by definition, the ring of matrices of the form

$$\begin{pmatrix} 1w_{11} & 0 & 0 \\ 0 & 1w_{\alpha\alpha} & \alpha w_{\alpha 2} \\ 0 & 0 & 2w_{22} \end{pmatrix}$$

where $w_{11}, w_{\alpha\alpha}, w_{\alpha 2}, w_{22} \in W$. Similarly, the ring $(S^* *_\sigma W)\# S^*$ consists of those matrices of the form

$$\begin{pmatrix} 1[k_1e_{11}] & 0 & 0 \\ 0 & 1[k_2e_{11}] & \alpha[k_3e_{22} + k_4e_{23}] \\ 0 & 0 & 2[k_5e_{22} + k_6e_{23} + k_7e_{33}] \end{pmatrix}$$

where $k_i \in k$ for $1 \leq i \leq 7$, the matrix units e_{ij} are taken from the 3×3 matrix ring over k, and we have listed the elements of S^* in the order $1, \alpha, 2$. The isomorphism $\Theta : \langle \widehat{S}^* *_{\widehat{\sigma}} W\rangle \to \langle S^* *_\sigma W\rangle \# S^*$ and its restriction $\widehat{S}^* *_{\widehat{\sigma}} W \to (S^* *_\sigma W)\# S^*$ are then precisely the expected ones, given the representations of $\langle \widehat{S}^* *_{\widehat{\sigma}} W\rangle$ and $\widehat{S}^* *_{\widehat{\sigma}} W$ provided in Example 2.2. □

In a manner analogous to that described above for actions, we get the following result for reversing actions. The proof is quite similar to the one given in Theorem 2.3, and hence is omitted.

Theorem 2.5. *Let S be a semigroup, and let γ be a reversing action of S^* as endomorphisms on the ring A. Then there is an isomorphism of rings*

$$\langle A *_{\widehat{\gamma}} \widehat{S}^*\rangle \cong \langle A *_\gamma S^*\rangle \# S^*,$$

given by the linear extension of $a(s,x) \longmapsto ase_{sx,x}$. If in addition A is unital then this isomorphism restricts to an isomorphism $A *_{\widehat{\gamma}} \widehat{S}^* \cong (A *_\gamma S^*)\# S^*$. Moreover, if S is l.i. and γ is locally unital, then each of these latter two rings is unital, and the isomorphism is as unital rings.

Theorems 2.3 and 2.5 accomplish one of the goals described in the introduction. Specifically, we have constructed a new semigroup from an existing one, and shown that skew–smash rings may be realized as skew semigroup rings for these new semigroups. In the remainder of this section we point out some specific consequences of these two theorems.

We begin by noting that there is a strong connection between the semigroup $\widehat{S}$ and certain preordered sets, described as follows. Let S be any semigroup. We define a relation $\leq$ on S^* by setting, for each pair $g,h \in S^*$, $g \leq h$ if there exists $f \in S^*$ with $g = fh$. Then $\leq$ is easily shown to be transitive. Moreover, if S is l.i., then $\leq$ is a preorder on S^*. This preorder on S^* produces the semigroup $S^{\leq}$, consisting of elements of the form $\leq_{fh,h}$ where $f,h \in S^*$ having $fh \in S^*$. When S is right $*$-cancellative, it is easy to show that $\widehat{S}$ is isomorphic to $S^{\leq}$, by the map $(s,x) \mapsto \leq_{sx,x}$.

Next, let $\gamma : S^* \to E(A)$ be a reversing action of S^* as endomorphisms on A. We define the function Ψ: $A\widehat{S}^* \to A *_{\widehat{\gamma}} \widehat{S}^*$ as the linear extension of $(a(s,x))\Psi = a^{(sx)\gamma}(s,x)$. It is straightforward to show that Ψ is a ring homomorphism. Moreover, if $\gamma: S^* \to Aut(A)$ is a reversing action of S as automorphisms on A, then Ψ is clearly bijective. Furthermore, if S is right $*$-cancellative and l.i. then the semigroup ring $A\widehat{S}^*$ is precisely the incidence ring $I(S^*,A)$ with the ordering on S^* given in the previous paragraph. Thus we have

Corollary 2.6. *If $\gamma: S^* \to Aut(A)$ is a reversing action of the semigroup S as automorphisms on A, then there is an isomorphism of rings*

$$A\widehat{S}^* \cong \langle A *_{\widehat{\gamma}} \widehat{S}^* \rangle \cong \langle A *_{\gamma} S^* \rangle \# S^*.$$

If A is unital, we get an isomorphism $A\widehat{S}^* \cong A *_{\widehat{\gamma}} \widehat{S}^* \cong (A *_{\gamma} S^*) \# S^*$, as these three rings are equal to the three rings (respectively) in the above displayed sequence of isomorphisms. Moreover, if S is right $*$-cancellative and l.i. then each of these rings is isomorphic to the incidence ring $I(S^*,A)$.

A construction analogous to the one described prior to the above corollary can be performed in situations where $\sigma : S^* \to E(A)$ is an action of S^* on A as endomorphisms; this yields a morphism from $I(S^*,A)^{\mathrm{op}}$ to $\widehat{S}^{o*} *_{\widehat{\sigma}} A$, which we do not investigate further here. However, in case $\sigma : S^* \to Aut(A)$ is an action of S^* as automorphisms on A, then by considering the linear extension of the function from $\widehat{S}^*A$ to $\widehat{S}^* *_{\widehat{\sigma}} A$ which takes $(s,x)a \longmapsto (s,x)a^{(x)\sigma^{-1}}$, a computation similar to that given above yields

Corollary 2.7. *If $\sigma: S^* \to Aut(A)$ is an action of the semigroup S as automorphisms on A, then there is an isomorphism of rings*

$$\widehat{S}^*A \cong \langle \widehat{S}^* *_{\widehat{\sigma}} A \rangle \cong \langle S^* *_{\sigma} A \rangle \# S^*$$

If A is unital, we get an isomorphism $\widehat{S}^*A \cong \widehat{S}^* *_{\widehat{\sigma}} A \cong (S^* *_{\sigma} A) \# S^*$, as these three rings are equal to the three rings (respectively) in the above displayed sequence of *isomorphisms. Moreover, if S is right $*$-cancellative and l.i. then each of these rings is isomorphic to the incidence ring $I(S^*, A)$.*

As another consequence of Theorems 2.3 and 2.5, we use them along with [4, Corollary 2.27 and Proposition 3.6(3)] to obtain immediately

Corollary 2.8. *Let A be a unital ring, and let S be an l.i. semigroup.*

(a) *Suppose that σ is a locally unital action of S^* as endomorphisms on A. Then the category $S^* *_\sigma A$-gr_* (consisting of those S-graded left $S^* *_\sigma A$-modules having $\{0\}$ z-component) is equivalent to the category $\widehat{S}^* *_{\widehat{\sigma}} A$-mod of all left $\widehat{S}^* *_{\widehat{\sigma}} A$-modules.*

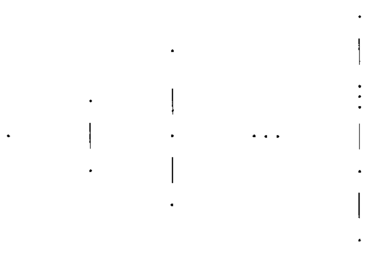
(b) *Suppose that γ is a locally unital reversing action of S^* as endomorphisms on A. Then the category $A *_\gamma S^*$-gr_* (consisting of those S-graded left $A *_\gamma S^*$-modules having $\{0\}$ z-component) is equivalent to the category $A *_{\widehat{\gamma}} \widehat{S}^*$-mod of all left $A *_{\widehat{\gamma}} \widehat{S}^*$-modules.*

(c) *If S is right $*$-cancellative, and if σ is an action of S^* as automorphisms on A, or if γ is a reversing action of S^* as automorphisms on A, then the categories $S^* *_\sigma A$-gr_* and $A *_\gamma S^*$-gr_* are each equivalent to the category $I(S^*, A)$-mod of all left modules over the incidence ring $I(S^*, A)$.*

Example 2.9. Let B be any unital ring. Let S denote the semigroup of Γ, where Γ is the directed graph

$$\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longrightarrow \cdots \longrightarrow \underset{n}{\cdot}$$

The path ring $R = B\Gamma$ is equal to the (contracted) semigroup ring BS^*, so that R is naturally graded by S. A straightforward computation shows that the preordered set $S^{\leq}$ is actually partially ordered. In fact, this partially ordered set may be viewed as a disjoint union of chains, represented as a Hasse diagram by



Thus by Corollary 2.6 (using the identity action of S^* on B) we have an isomorphism

$$BS^* \# S^* \cong B\widehat{S}^* \cong I(S^*, B) \cong B \oplus U_2(B) \oplus U_3(B) \oplus \cdots \oplus U_n(B)$$

(direct sum as rings).

We note that by Corollary 2.8, the category BS^*-gr_* is in this way realized as the category of modules over a direct sum of incidence rings.

As a specific example, let S be the semigroup of Example 1.3. Then kS^* is the path algebra of the directed graph $_1\cdot \longrightarrow \cdot_2$, so that the ring $kS^*\#S^*$ is isomorphic to the ring $k \oplus U_2(k)$, which in turn is isomorphic to the ring W of Example 1.3. □

If G is a group, then for each g,h in G we have $g \le h$ in the preordering described above, as $g = (gh^{-1})h$. In particular, for any ring A and finite group G the ring $I(G,A)$ is equal to $M_{|G|}(A)$, the full $|G| \times |G|$ matrix ring with coefficients in A. Thus by Corollaries 2.6 and 2.7 we immediately obtain the following "duality" results of Cohen and Montgomery.

Corollary 2.10. ([6, Theorem 3.2]) *Let G be a finite group. If σ is an action of G as automorphisms on the unital ring A, then*

$$(G *_\sigma A)\# G \cong M_{|G|}(A).$$

If γ is a reversing action of G as automorphisms on the unital ring A, then

$$(A *_\gamma G)\# G \cong M_{|G|}(A).$$

3. "Skew–smash–skew" constructions

Unlike the results of the previous section (especially Theorems 2.3 and 2.5), and unlike the results for groups (e.g. [6, Theorem 3.5]), we will not in general be able to realize "smash–skew" rings as skew semigroup rings with coefficients in the original graded ring. Perhaps surprisingly, however, we *are* able to obtain satisfactory results in the situation where the underlying graded ring is itself a skew semigroup ring. In fact, in this setting we obtain results which are similar in flavor to those of the skew–smash variety. Specifically, the underlying semigroup is used to produce new, related semigroups, and the resulting skew–smash–skew rings are then shown to be isomorphic to skew semigroup rings over these new semigroups.

The process which yields the more general type of skew semigroup rings (i.e. those of the form $\langle S^* * A\rangle$ and $\langle A * S^*\rangle$) will produce skew–smash–skew rings which are rather straightforward to describe; see e.g. Propositions 3.2 and 3.3. Such a description will follow from the results of Section 2, together with some observations about "skew–skew" rings of a special type. On the other hand, the skew semigroup ring construction which produces rings of the form $S^* * A$ and $A * S^*$ for unital A will yield skew–smash–skew rings which are significantly different from these, and are not as easy to describe. Thus most our focus in this section will be on rings of these latter two types.

Rephrasing, our goal for this section is to describe rings of the form skew–smash–skew. There are two types of skew semigroup rings; one general, one arising only for unital coefficient rings. For each of these, there are two types of resulting rings: those arising from actions and those arising from reversing actions. Both types are graded by the appropriate semigroup, so we can form the corresponding skew–smash rings, as was described in the previous section. But as noted in Definition 1.4, there are four naturally occurring smash–skew rings. As a consequence, there are $2 \times 2 \times 4 = 16$ types of skew–smash–skew rings which we aim to investigate. Clearly an explicit description of all 16 types would exceed the limits of the readers' interest: thus we will, whenever possible, simply state the appropriate results, and indicate that the corresponding proof is either easy, or is completely analogous to a proof presented previously.

We begin by describing "skew–skew" rings of a special type.

Proposition 3.1. *Let A be a ring, and let S and T be semigroups. Suppose σ is an action of T^* as endomorphisms on A. Suppose further that κ is an action of S^* as endomorphisms on $\langle T^* *_\sigma A\rangle$ with the properties:*

(i) For each $s \in S^*$ and $t \in T^*$ either $(tA)^{(s)\kappa} = 0$ or $(tA)^{(s)\kappa} \subseteq t_1 A$ for some $t_1 \in T^*$. In this case we denote the (necessarily unique) element t_1 by $t^{(s)\kappa}$.

(ii) $(t)\sigma = (t^{(s)\kappa})\sigma$ whenever $t \in T^*$ has $(tA)^{(s)\kappa} \subseteq t^{(s)\kappa}A$.

In this situation we let $U = S^* \times T^* \cup \{z_U\}$, where z_U is some symbol not in $S^* \times T^*$; U is a semigroup with multiplication defined by setting z_U to be the zero element of U, and

$$(s,t)\cdot(s',t') = \begin{cases} (ss', t^{(s')\kappa}t') & \text{if } ss' \in S^* \text{ and } t^{(s')\kappa}t' \in T^*; \\ z_U & \text{otherwise.} \end{cases}$$

(So U is just the semidirect product of S and T via κ). Then the map $\theta : U^* \to E(A)$ given by setting $((s,t))\theta = (t)\sigma$ is an action of U^* as endomorphisms on A such that $\langle S^* *_\kappa \langle T^* *_\sigma A\rangle\rangle \cong \langle U^* *_\theta A\rangle$. That is, the "skew–skew" ring $\langle S^* *_\kappa \langle T^* *_\sigma A\rangle\rangle$ is isomorphic to a skew semigroup ring with coefficients in A.

Proof. All assertions in this proof are straightforward to check; their verifications are left to the reader. We note that property (ii) is required to show that θ is an action, and that the indicated isomorphism $\langle S^* *_\kappa \langle T^* *_\sigma A\rangle\rangle \mapsto \langle U^* *_\theta A\rangle$ is given by the linear extension of $s[ta] \mapsto (s,t)a$. $\square$

Proposition 3.2. *Let σ be an action of the right $*$-cancellative semigroup S as endomorphisms on the ring A. Then there exists a semigroup $\underline{S}$ and an action θ of $\underline{S}^*$ as endomorphisms on A such that*

$$\langle S^* *_\rho \langle S^* *_\sigma A\rangle \# S^*\rangle \cong \langle \underline{S}^* *_\theta A\rangle .$$

That is, the skew–smash–skew ring $\langle S^* *_\rho \langle S^* *_\sigma A\rangle \# S^*\rangle$ is isomorphic to a skew semigroup ring with coefficients in A.

Proof. By Theorem 2.3 we have that $\langle S^* *_\sigma A\rangle \# S^* \cong \langle \widehat{S}^* *_{\widehat{\sigma}} A\rangle$. It is now easy to check that under this identification, the action ρ of S^* on $\langle S^* *_\sigma A\rangle \# S^*$ given in

Definition 1.4 corresponds to an action of S^* on $\widehat{S}^* *_{\widehat{\sigma}} A$ (which we also denote by ρ) defined by the linear extension of

$$((s,x)a)^{(y)\rho} = \begin{cases} (s,xy)a & \text{if } sxy \in S^*; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, ρ and $\widehat{\sigma}$ are easily shown to satisfy properties (i) and (ii) of Proposition 3.1, with ρ in the role of κ and $\widehat{\sigma}$ in the role of σ. Thus Proposition 3.1 applies, and yields $\langle S^* *_\rho \langle S^* *_\sigma A\rangle \# S^*\rangle \cong \langle S^* *_\rho \langle \widehat{S}^* *_{\widehat{\sigma}} A\rangle\rangle \cong \langle U^* *_\theta A\rangle$. We denote U in this situation by $\underline{S}$; $\underline{S}$ can be explicitly described as $\underline{S} = S^* \times \widehat{S}^* \cup \{z_{\underline{S}}\} = \{(y,(s,x)) \mid sx \in S^*\} \cup \{z_{\underline{S}}\}$, where multiplication is given by

$$\text{in } \underline{S}: \quad (y,(s,x)) \cdot (y',(s',x')) = \begin{cases} (yy',(ss',x')) & \text{if } yy' \in S^*,\ ss'x' \in S^* \\ & \quad \text{and } xy' = s'x'; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the isomorphism $\langle S^* *_\rho \langle S^* *_\sigma A\rangle \# S^*\rangle \cong \langle \underline{S}^* *_\theta A\rangle$ is given by $\underline{\Omega} : s'[(sa)e_{sx,x}] \mapsto (s',(s,x))a$. □

There are seven additional skew–smash–skew rings which can be formed using the general version of skew semigroup rings; their development is completely analogous to the one given above. Specifically, we consider an action or reversing action of a semigroup S on a ring of the form $\langle T^* * A\rangle$ or $\langle A * T^*\rangle$, and describe the resulting skew–skew ring as a skew semigroup ring with coefficients in A. Then, employing the identifications afforded by Theorems 2.3 and 2.5 we describe an isomorphism from an appropriate skew–smash–skew ring to the resulting skew semigroup ring. We describe three of these rings in the next proposition. The proofs are entirely similar to the proofs given in Propositions 3.1 and 3.2 and so are omitted. (We remark that the right $*$-cancellativity of S is required to produce the action ρ of S^* on smash product rings. On the other hand, even though this condition is not required to produce the reversing action λ of S^* on smash product rings, due to our desire to mimic property (i) in Proposition 3.1, we do need right $*$-cancellativity in order to realize skew–smash–skew rings for reversing actions as skew semigroup rings for new semigroups.)

Proposition 3.3. *Let A be a ring, and let S and T be semigroups. Suppose σ is an action or reversing action of T^* as endomorphisms on A. Suppose further that κ is an action or reversing action of S^* as endomorphisms on $\langle T^* *_\sigma A\rangle$ (resp. $\langle A *_\sigma T^*\rangle$) with the properties:*

(i) *For each $s \in S^*$ and $t \in T^*$ either $(tA)^{(s)\kappa} = 0$ (resp. $(At)^{(s)\kappa} = 0$) or $(tA)^{(s)\kappa} \subseteq t_1 A$ (resp. $(At)^{(s)\kappa} \subseteq At_1$) for some $t_1 \in T^*$. In this case we denote the (necessarily unique) element t_1 by $t^{(s)\kappa}$.*

(ii) *$(t)\sigma = (t^{(s)\kappa})\sigma$ whenever $t \in T^*$ has $(tA)^{(s)\kappa} \subseteq t^{(s)\kappa}A$ (resp. $(At)^{(s)\kappa} \subseteq At^{(s)\kappa}$).*

(1) *In case σ is an action and κ is a reversing action then there exists a semigroup U and an action θ of U^* on A such that $\langle\langle T^* *_\sigma A\rangle *_\kappa S^*\rangle \cong \langle U^* *_\theta A\rangle$. If S is right $*$-cancellative, then applying this general result with $\widehat{S}$ in the role of*

T and S^{op} in the role of S together with Theorem 2.3 yields an isomorphism $\underset{\sim}{\Omega} : \langle ((\langle S^* *_\sigma A\rangle \# S^*) *_\rho S^{o*}\rangle \cong \langle \underset{\sim}{S}^* *_\theta A\rangle$, where $\underset{\sim}{S}^* = \widehat{S}^* \times S^{o*}$ as sets, and multiplication is given by:

$$\text{in } \underset{\sim}{S}: \quad ((s,x),y)\cdot((s',x'),y') = \begin{cases} ((ss',x'y),y'y) & \text{if } y'y \in S^*, ss'x'y \in S^* \\ & \quad \text{and } x = s'x'y; \\ z_{\underset{\sim}{S}} & \text{otherwise.} \end{cases}$$

(2) *In case σ and κ are each reversing actions then there exists a semigroup U and a reversing action θ of U^* on A such that $\langle\langle A *_\sigma T^*\rangle *_\kappa S^*\rangle \cong \langle A *_\theta U^*\rangle$. If S^* is right $*$-cancellative, then applying this general result with $\widehat{S}$ in the role of T together with Theorem 2.5 yields an isomorphism $\underset{\rightarrow}{\Omega} : \langle ((\langle A *_\sigma S^*\rangle \# S^*) *_\lambda S^*\rangle \cong \langle A *_\theta \underset{\rightarrow}{S}^*\rangle$, where $\underset{\rightarrow}{S}^* = \widehat{S}^* \times S^*$ as sets, and multiplication is given by:*

$$\text{in } \underset{\rightarrow}{S}: \quad ((s,x),y)\cdot((s',x'),y')$$

$$= \begin{cases} ((ss',k),yy') & \text{if there exists (unique) } k \in S^* \text{ with} \\ & \quad ky = x' \text{ and } x = s'k, \text{ and } yy' \in S^*; \\ z_{\underset{\rightarrow}{S}} & \text{otherwise.} \end{cases}$$

(3) *In case σ is a reversing action and κ is an action then there exists a semigroup U and a reversing action θ of U^* on A such that $\langle S^* *_\kappa \langle A *_\sigma T^*\rangle\rangle \cong \langle A *_\theta U^*\rangle$. If S is right $*$-cancellative, then applying this general result with $\widehat{S}$ in the role of T and S^{op} in the role of S together with Theorem 2.5 yields an isomorphism $\underbrace{\Omega}$: $\langle S^{o*} *_\lambda ((\langle A *_\sigma S^*\rangle \# S^*)\rangle \cong \langle A *_\theta \underbrace{S}^*\rangle$, where $\underbrace{S}^* = S^{o*} \times \widehat{S}^{o*}$ as sets, and multiplication is given by*

$$\text{in } \underbrace{S}: \quad (y,(s,x))\cdot(y',(s',x')) = \begin{cases} (y'y,(ss',x')) & \text{if } y'y \in S^*,\ ss'x' \in S^* \\ & \quad \text{and } x = s'x'y'; \\ z_{\underbrace{S}} & \text{otherwise.} \end{cases}$$

We note that there are four additional statements analogous to those made in Propositions 3.2 and 3.3 above, for rings of the form $S^{o*} * A$ and $A * S^{o*}$; we invite the reader to provide such statements as he/she sees fit.

It is straightforward to check that the four semigroups $\underline{S}$, $\underset{\sim}{S}$, $\underset{\rightarrow}{S}$, and $\underbrace{S}$ are isomorphic in case S is a group.

We now undertake the task of describing skew–smash–skew rings in the context of unital rings. This is a rather messy undertaking in general. However, with a modicum of additional structure (namely, that S is l.i.), we can explicitly describe the elements of the "smash–skew" portion of such rings, which will in turn allow us to describe "skew–smash–skew" rings as skew semigroup rings for appropriate semigroups. As we shall see, there is a somewhat surprising lack of symmetry in these constructions. The information contained in the next lemma appears in [3, Section 4]; we omit the proof here.

Lemma 3.4. *Let R be a locally unital ring graded by the right ∗-cancellative l.i. semigroup S. Let ρ (resp. λ) be the action (resp. reversing action) described in Definition 1.4.*

(1) *The elements of $S^* *_\rho [R\# S^*]$ are sums of expressions of the form*

$$\{h[r_f e_{fg,g}] \mid fg \neq z,\ r_f \in R_f, \text{ and } fg = lh \text{ for some } l \in S^*\}.$$

(2) *The elements of $[R\# S^*] *_\rho S^{o*}$ are sums of expressions of the form*

$$\{[r_f e_{fg,g}]h \mid fg \neq z,\ r_f \in R_f, \text{ and } g = lh \text{ for some } l \in S^*\}.$$

(3) *The elements of $[R\# S^*] *_\lambda S^*$ are sums of expressions of the form*

$$\{[r_f e_{fg,g}]h \mid fg \neq z,\ r_f \in R_f, \text{ and } gh \neq z\}.$$

(4) *The elements of $S^{o*} *_\lambda [R\# S^*]$ are sums of expressions of the form*

$$\{h[r_f e_{fg,g}] \mid fg \neq z,\ r_f \in R_f, \text{ and } fgh \neq z\}.$$

The procedure carried out at the beginning of this section to describe general skew–smash–skew rings involved using the isomorphisms of Section 2, together with a description of skew–skew rings. We are unable to follow the same type of procedure here. Essentially, this is due to the fact that the process of multiplying scalar elements in $R\# S^*$ by an element of the form $(1_{R\# S^*})^{(s)\rho}$ or $(1_{R\# S^*})^{(s)\lambda}$ does not correspond to an endomorphism of $R\# S^*$. Thus we are required to treat each of these skew–smash–skew rings as a separate case, a process which will take up the remainder of this article.

As long as the action or reversing action of the semigroup S on a ring of the form $\langle T^* *_\sigma A\rangle$ or $\langle A *_\sigma T^*\rangle$ restricts to an action or reversing action of S on subrings of the form $T^* *_\sigma A$ or $A *_\sigma T^*$, then the resulting skew–smash–skew ring in the unital setting will be a subring of the corresponding general skew–smash–skew process. We have already exhibited isomorphisms between these general skew–smash–skew rings and skew semigroup rings with coefficients in A (Propositions 3.2 and 3.3). Fortunately, we will be able to utilize restrictions of these isomorphisms to the appropriate subrings in order to realize these subrings as skew semigroup rings with coefficients in A.

We now give the first of eight results in which we realize a skew–smash–skew ring in the unital setting as a skew semigroup ring with coefficients in the original ring of scalars.

Theorem 3.5. *Let S be a right ∗-cancellative l.i. semigroup, and let $\sigma : S^* \to E(A)$ be a locally unital action of S^* as endomorphisms on A. Then there is an l.i. semigroup $\overline{S}$ and a locally unital action $\overline{\sigma} : \overline{S}^* \to E(A)$ which yields an isomorphism of rings*

$$\overline{S}^* *_{\overline{\sigma}} A \cong S^* *_\rho [(S^* *_\sigma A)\# S^*].$$

Proof. We let $\overline{S}$ denote the l.i. semigroup whose elements are

$$\overline{S} = \{(l,h,f,g) \in (S^*)^4 \mid fh \in S^*,\ lg \in S^*, \text{ and } fh = lg\} \cup \{\overline{z}\},$$

and where multiplication is defined in $\overline{S}$ by setting

$$\text{in } \overline{S}: \quad (l,h,f,g)\cdot(l',h',f',g') = \begin{cases} (ll',hh',f,g') & \text{if } ll'g' \in S^*,\ fhh' \in S^* \\ & \text{and } g = f'; \\ \overline{z} & \text{otherwise.} \end{cases}$$

It is easy to check that $\overline{E} = \{(e_g, e'_g, g, g) \mid g \in S^*\}$ is a set of local identities for $\overline{S}$, and that $\overline{S}$ is a category whenever S is. We define the locally unital action $\overline{\sigma} : \overline{S}^* \to E(A)$ of $\overline{S}$ as endomorphisms on A by setting $(l,h,f,g)\overline{\sigma} = (l)\sigma$.

A tedious check (utilizing the right $*$-cancellativity of S) reveals that for $l,g,h \in S^*$ with $lg \in S^*$,

$$(1_{R\#S^*})^{(h)\rho} \cdot l[1^{(l)\sigma}a]e_{lg,g} = \begin{cases} l[1^{(l)\sigma}a]e_{lg,g} & \text{if there exists } f \in S^* \text{ with } lg = fh; \\ 0 & \text{otherwise} \end{cases}$$

as elements of $(S^* *_\sigma A)\# S^*$.

For the remainder we utilize the notation given in the proof of Proposition 3.2. We define the function $\varsigma : \overline{S} \to \underline{S}$ by $\varsigma : (l,h,f,g) \mapsto (h,(l,g))$. It is tedious to show that ς is an injective semigroup homomorphism (the right $*$-cancellativity of S is required for injectivity). This function is easily shown to induce an injective ring homomorphism $\overline{\varsigma} : \overline{S}^* *_{\overline{\sigma}} A \to \langle \underline{S}^* *_\theta A\rangle$. We now consider the map $\overline{\Omega} : \overline{S}^* *_{\overline{\sigma}} A \to S^* *_\rho [(S^* *_\sigma A)\# S^*]$ defined by setting $\overline{\Omega} = \overline{\varsigma} \circ \underline{\Omega}^{-1}$. Specifically, $\overline{\Omega}$ is the linear extension of the function described by

$$\overline{\Omega} : (l,h,f,g)[1^{(l,h,f,g)\overline{\sigma}}a] \longmapsto h[l[1^{(l)\sigma}a]e_{lg,g}].$$

The computation above indicates that in this situation we have $h[l[1^{(l)\sigma}a]e_{lg,g}] = h[(1_{R\#S^*})^{(h)\rho} \cdot l[1^{(l)\sigma}a]e_{lg,g}]$, so that $\overline{\Omega}$ indeed maps into $S^* *_\rho [(S^* *_\sigma A)\# S^*]$. That $\overline{\Omega}$ is onto is seen by using the same computation. □

In a manner analogous to that described above, we may use the semigroup $\overline{S}$ to obtain a description of another skew–smash–skew ring.

Theorem 3.6. *Let S be a finite right $*$-cancellative l.i. semigroup, and let $\gamma : S^* \to E(A)$ be a locally unital reversing action of S as endomorphisms on A. Then there is an isomorphism of rings $A *_{\overline{\gamma}} \overline{S}^* \cong S^* *_\rho [(A *_\gamma S^*)\# S^*]$.*

Proof. We define the map $\overline{\Omega} : A *_{\overline{\gamma}} \overline{S}^* \to S^* *_\rho [(A *_\gamma S^*)\# S^*]$ to be the linear extension of the function $[a1^{(l,h,f,g)\overline{\gamma}}](l,h,f,g) \longmapsto h[[a1^{(l)\gamma}]le_{lg,g}]$. By proceeding in a manner similar to that given in the proof of the previous theorem, it is straightforward to show that $\overline{\Omega}$ is indeed an isomorphism of rings. □

We now give three results analogous to Theorem 3.5 for three additional types of skew–smash–skew rings.

Theorem 3.7. *Let S be a right $*$-cancellative l.i. semigroup, and let $\sigma : S^* \to E(A)$ be a locally unital action of S^* as endomorphisms on A.*

(1) *There is an l.i. semigroup $\widetilde{S}$ and a locally unital action $\widetilde{\sigma} : \widetilde{S}^* \to E(A)$ which yields an isomorphism of rings $\widetilde{S}^* *_{\widetilde{\sigma}} A \cong [(S^* *_\sigma A)\# S^*] *_\rho S^{o*}$.*

(2) *There is an l.i. semigroup $\overrightarrow{S}$ and a locally unital action $\overrightarrow{\sigma} : \overrightarrow{S}^* \to E(A)$ which yields an isomorphism of rings $\overrightarrow{S}^* *_{\overrightarrow{\sigma}} A \cong [(S^* *_\sigma A)\# S^*] *_\lambda S^*$.*

(3) *Using the same semigroup $\widetilde{S}$ and same action $\widetilde{\sigma}$ as described in statement (1), there is an isomorphism of rings $\widetilde{S}^* *_{\widetilde{\sigma}} A \cong S^{o*} *_\lambda [(S^* *_\sigma A)\# S^*]$.*

Proof. As the verifications of the statements made in this proof are analogous to those made in the proof of Theorem 3.5, we simply furnish the appropriate definitions of the new semigroups, actions, and isomorphisms. For brevity we utilize the notation given in the statement of Proposition 3.3.

(1) We let $\widetilde{S}$ denote the l.i. semigroup whose elements are

$$\widetilde{S} = \{(f,g,l) \in (S^*)^3 \mid fgl \in S^*\} \cup \{\widetilde{z}\},$$

where multiplication is defined in $\widetilde{S}$ by setting

$$\text{in } \widetilde{S}: \quad (f,g,l)\cdot(f',g',l') = \begin{cases} (ff',g',l'l) & \text{if } g = f'g'l'; \\ \widetilde{z} & \text{otherwise.} \end{cases}$$

The set $\widetilde{E} = \{(e_g, g, e'_g) \mid g \in S^*\}$ is a set of local identities for $\widetilde{S}$, and $\widetilde{S}$ is a category whenever S is. We define $\widetilde{\sigma} : \widetilde{S}^* \to E(A)$ by setting $(f,g,l)\widetilde{\sigma} = (f)\sigma$. We now define the function $\varsigma : \widetilde{S} \to S$ by setting $(f,g,l) \mapsto ((f,gl),l)$. It is tedious to show that ς is an injective semigroup homomorphism (the right $*$-cancellativity of S is required for injectivity). This function induces an injective ring homomorphism $\widetilde{\varsigma}$: $\widetilde{S}^* *_{\widetilde{\sigma}} A \to \langle S^* *_\theta A\rangle$. The map $\widetilde{\Omega} : \widetilde{S}^* *_{\widetilde{\sigma}} A \to [(S^* *_\sigma A)\# S^*] *_\rho S^{o*}$ produced by setting $\widetilde{\Omega} = \widetilde{\varsigma} \circ \Omega^{-1}$ is the desired isomorphism. Specifically, $\widetilde{\Omega}$ is the linear extension of the function described by

$$\widetilde{\Omega} : (f,g,l)[1^{(f,g,l)\widetilde{\sigma}}a] \longmapsto [f[1^{(f)\sigma}a]e_{fgl,gl}]l.$$

(2) We let $\overrightarrow{S}$ denote the l.i. semigroup whose elements are

$$\overrightarrow{S} = \{(f,h,l) \in (S^*)^3 \mid fh \in S^* \text{ and } hl \in S^*\} \cup \{\overrightarrow{z}\},$$

where multiplication is defined in $\overrightarrow{S}$ by setting

$$\text{in } \overrightarrow{S}: \quad (f,h,l)\cdot(f',h',l') = \begin{cases} (ff',k,ll') & \text{if } ff' \in S^*, ll' \in S^*, \text{ and } \exists k \in S^* \\ & \quad \text{with } h = f'k \text{ and } kl = h'; \\ \overrightarrow{z} & \text{otherwise.} \end{cases}$$

The set $\overrightarrow{E} = \{(e_g, g, e'_g) \mid g \in S^*\}$ is a set of local identities for $\overrightarrow{S}$. Unlike the semigroups $\overline{S}$ and $\widetilde{S}$, the semigroup $\overrightarrow{S}$ need not be a category even when S is (see the remarks subsequent to this proof). We define $\overrightarrow{\sigma} : \overrightarrow{S}^* \to E(A)$ by setting $(f,h,l)\overrightarrow{\sigma} = (f)\sigma$. We now define the function $\varsigma : \overrightarrow{S} \to \underset{\rightarrow}{S}$ by setting $(f,h,l) \mapsto ((f,h),l)$. It is tedious to show that ς is an injective semigroup homomorphism. This function induces an injective ring homomorphism $\overrightarrow{\varsigma} : \overrightarrow{S}^* *_{\overrightarrow{\sigma}} A \to \langle \overrightarrow{S}^* *_\theta A \rangle$. The map $\overrightarrow{\Omega} : \overrightarrow{S}^* *_{\overrightarrow{\sigma}} A \to [(S^* *_\sigma A)\# S^*] *_\lambda S^*$ produced by setting $\overrightarrow{\Omega} = \widetilde{\varsigma} \circ \underset{\rightarrow}{\Omega}^{-1}$ is the desired isomorphism. Specifically, $\overrightarrow{\Omega}$ is the linear extension of the function described by

$$\overrightarrow{\Omega} : (f,h,l)[1^{(f,h,l)\overrightarrow{\sigma}} a] \longmapsto [f[1^{(f)\sigma} a] e_{fh,h}] l.$$

(3) If R is any locally unital ring graded by S, then by [3, Proposition 4.6(1)] we have $S^{o*} *_\lambda [R \# S^*] \cong [R\# S^*] *_\rho S^{o*}$. The result now follows by setting $R = S^* *_\sigma A$, and using part (1) of this theorem. For completeness, we note that an explicit description of this isomorphism is given by the linear extension of the function

$$\widehat{\Omega} : (f,g,l)[1^{(f,g,l)\widetilde{\sigma}} a] \longmapsto l[f[1^{(f)\sigma} a] e_{fg,g}]. \qquad \square$$

Unlike the corresponding results for $\overline{S}$ and $\widetilde{S}$, it turns out that $\overrightarrow{S}$ need not be a category when S is a category. For instance, let X be the totally ordered set $\{a,b,c,d\}$ where $a \le b \le c \le d$. We let S denote the semigroup $X^{\le}$ as described in Section 2; it is easy to show that S is a category. For notational simplicity we set $\le_{a,b} = \alpha$, $\le_{b,c} = \beta$, $\le_{c,d} = \gamma$, $\le_{a,a} = 1$, $\le_{b,b} = 2$, $\le_{c,c} = 3$, and $\le_{d,d} = 4$. Then it is straightforward to check that in $\overrightarrow{S}$ we have

$$(\alpha,\beta,\gamma)\cdot(\beta,\gamma,4) = (\alpha\beta,3,4) \ne \overrightarrow{z} \quad \text{and} \quad (\beta,\gamma,4)\cdot(\gamma,4,4) = (\beta\gamma,4,4) \ne \overrightarrow{z},$$

$$\text{while} \quad [(\alpha,\beta,\gamma)\cdot(\beta,\gamma,4)]\cdot(\gamma,4,4) = (\alpha\beta,3,\gamma)\cdot(\gamma,4,4) = \overrightarrow{z}.$$

Suppose that the semigroup S is *super-cancellative*; that is, if $u,u',v,w,w' \in S^*$ with $uvw = u'vw' \ne z$, then $u = u'$ and $w = w'$. For instance, any semigroup arising from a partially ordered set or an acyclic directed graph has this property. We define a relation τ on S^* by setting, for each pair $c,d \in S^*$,

$$c\tau d \quad \text{in case} \quad \exists f,k,g \in S^* \text{ with } c = fk \text{ and } d = kg.$$

Then (as described in [1]) we may form the "generalized incidence ring" $I(S^*,\tau,A)$. A straightforward check yields that, in this setting, $I(S^*,\tau,A)$ is isomorphic to the semigroup ring $\overrightarrow{S}^* A$, via the linear extension of the map which takes $(f,h,l)a \in \overrightarrow{S}^* A$ to $ae_{fh,hl} \in I(S^*,\tau,A)$. In fact, the skew–smash–skew construction described in this section was the first author's original motivation for considering these generalized incidence rings.

In the situation where R is an algebra over a field which is graded by the semigroup S, Beattie [5, Example 2.10] has given a description of rings of the form $[R\# S^*] *_\lambda S^*$

as rings of matrices with a nonstandard multiplication. Her approach utilizes notations and techniques from Hopf algebras. Beattie's description of $[R\#S^*]*_\lambda S^*$ is *not* as a skew semigroup ring with coefficients in R, but rather as matrices whose entries are taken from appropriate S-components of R. In the specific case where σ is an action as automorphisms and $R = S^* *_\sigma A$, then each of the S-graded components of R is isomorphic to A. This observation yields that when A is an algebra over a field, our construction coincides with Beattie's.

We complete our description of skew–smash–skew rings by listing out the final three isomorphisms; these are the analogs of Theorem 3.6.

Theorem 3.8. *Let S be a finite right $*$-cancellative l.i. semigroup, and let $\gamma : S^* \to E(A)$ be a locally unital reversing action of S^* as endomorphisms on A.*

(1) *There is an isomorphism of rings $A *_{\widetilde{\gamma}} \widetilde{S}^* \cong [(A *_\gamma S^*)\#S^*] *_\rho S^{o*}$, given by the linear extension of the function $[a1^{(f,g,l)\widetilde{\gamma}}](f,g,l) \longmapsto [[a1^{(f)\gamma}]fe_{fgl,gl}]l$.*

(2) *There is an isomorphism of rings $A *_{\overrightarrow{\gamma}} \overrightarrow{S}^* \cong [(A *_\gamma S^*)\#S^*] *_\lambda S^*$, given by the linear extension of the function $[a1^{(f,h,l)\overrightarrow{\gamma}}](f,h,l) \longmapsto [[a1^{(f)\gamma}]fe_{fh,h}]l$.*

(3) *There is an isomorphism of rings $A *_{\widetilde{\gamma}} \widetilde{S} \cong S^{o*} *_\lambda [(A *_\gamma S^*)\#S^*]$, given by the linear extension of the function $[a1^{(f,g,l)\widetilde{\gamma}}](f,g,l) \longmapsto l[[a1^{(f)\gamma}]fe_{fg,g}]$.*

We now describe the semigroups $\overline{S}$, $\widetilde{S}$, and $\overrightarrow{S}$ for a particular semigroup S.

Example 3.9. We again let S denote the semigroup $\{1, \alpha, 2, z\}$ described in the previous Examples. It is tedious but straightforward to verify the following statements regarding the semigroups which arise from S.

(1) $\overline{S}^* = \{(1,1,1,1), (1,\alpha,1,\alpha), (1,2,\alpha,\alpha), (\alpha,\alpha,1,2), (\alpha,2,\alpha,2), (2,2,2,2)\}$. Moreover, $\overline{S}$ is isomorphic to the semigroup with zero consisting of the upper triangular 3×3 matrix units and 0. Specifically,

$$\overline{S} \cong \{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}, 0\}.$$

(2) $\widetilde{S}^* = \{(1,1,1), (1,1,\alpha), (1,\alpha,2), (\alpha,2,2), (2,2,2)\}$. Moreover, $\widetilde{S}$ is isomorphic to the semigroup with zero consisting of the following five 3×3 matrix units and 0. Specifically,

$$\widetilde{S} \cong \{e_{11}, e_{21}, e_{22}, e_{23}, e_{33}, 0\}.$$

(3) $\overrightarrow{S}^* = \{(1,1,1), (1,1,\alpha), (1,\alpha,2), (\alpha,2,2), (2,2,2)\}$. Moreover, $\overrightarrow{S}$ is isomorphic to the semigroup with zero which arises from $\overline{S}$ by identifying e_{13} with 0. Specifically,

$$\overrightarrow{S} \cong \{e_{11}, e_{12}, e_{22}, e_{23}, e_{33}, 0\} \quad \text{where we define} \quad e_{12} \cdot e_{23} = 0.$$

We note that, as S is a category, the elements of $\widetilde{S}^*$ and $\overrightarrow{S}^*$ coincide. However, as is apparent from their representations as matrix units, the semigroups $\widetilde{S}$ and $\overrightarrow{S}$ are not isomorphic. □

We conclude with some observations regarding the constructions presented in this section. First, we note that the descriptions given in the above Example indicate that certain "related" skew semigroup rings need not be isomorphic. Specifically, we let S act on the field k via the identity action, and we let A denote the ring $kS^* \# S^*$. Then by invoking the appropriate theorems of this section, the rings $S^* *_\rho A$, $A *_\rho S^{o*}$, and $A *_\lambda S^*$ are isomorphic, respectively, to the semigroup algebras $k\overline{S}^*, k\widetilde{S}^*$, and $k\overrightarrow{S}^*$. It is straightforward to show that these three semigroup algebras are pairwise nonisomorphic.

Second, if G is a group then we define the semigroup $T_G = \{(f,g,h) \in G^3\} \cup \{z_T\}$, where multiplication in T_G is given by setting $(f,g,h) \cdot (f',g',h') = (f,g',hh')$ if $g = f'$, and z_T otherwise. Then the semigroups $\overline{G}, \widetilde{G}$, and $\overrightarrow{G}$ are each isomorphic to T_G; the appropriate isomorphisms to T_G are given by sending (l,h,f,g) in $\overline{G}$ to (f,g,h), sending (f,g,l) in $\widetilde{G}$ to (fgl,g,l^{-1}), and sending (f,h,l) in $\overrightarrow{G}$ to (fh,hl,f). Moreover, for any ring A and any action σ (resp. reversing action γ) of G as automorphisms on A, each of the resulting skew–smash–skew rings is isomorphic to the full $|G| \times |G|$ matrix ring $M_{|G|}(G *_\sigma A)$ (resp. $M_{|G|}(A *_\gamma G)$). As a consequence of this last observation we conclude that, in contrast to the isomorphisms described in Corollaries 2.8 and 2.9, the skew semigroup rings involving $\overline{G}, \widetilde{G}$, and $\overrightarrow{G}$ need not be isomorphic to the corresponding (unskewed) semigroup rings.

Third, it is easy to show that if S is l.i., then S embeds in each of the semigroups $\overline{S}, \widetilde{S}$, and $\overrightarrow{S}$. These embeddings are *not* as direct summands.

Fourth, it is shown in [2] that if R is a locally unital ring graded by the l.i. semigroup S, then the ring $R\#S^*$ may be viewed as the ring $End_{R\text{-gr}}(U(R))$ of graded endomorphisms of the canonical S-graded module $U(R)$. As given in [2, Proposition 3.3], there are natural locally unital actions and locally unital reversing actions of S^* as endomorphisms on $End_{R\text{-gr}}(U(R))$. It is tedious but straightforward to show that these actions and reversing actions correspond to the actions and reversing actions of S^* on $R\#S^*$ given in Definition 1.4, under the identification of $R\#S^*$ with $End_{R\text{-gr}}(U(R))$.

Finally, suppose R is graded by the l.i. semigroup S. We would like to realize rings of the form $(R\#S^*) * S^*$ or $S^* * (R\#S^*)$ (i.e., general smash–skew rings) as skew semigroup rings with coefficients in R. This was done by Cohen and Montgomery for the case when $S = G$ is a finite group: by [6, Theorem 3.5] any ring of the form $(R\#G) * G$ or $G * (R\#G)$ is isomorphic $M_{|G|}(R)$, which in turn is isomorphic to $R\widehat{G}^*$. However, we are unable to obtain such a result for more general semigroups. Briefly, this is because for a group G, for any two elements $c,d \in G$, the number of elements in the semigroup $\overline{G}$ (described in the proof of Theorem 3.5) of the form (c,h,d,g) is exactly $|G|$. In an arbitrary semigroup, however, this number can vary (depending on the pair c,d).

On the other hand, the isomorphisms presented in this section indicate that we *are* able to obtain results similar to those described in the previous paragraph in the specific case when R itself is a skew semigroup ring over S having coefficients in the ring A. In this situation, however, we obtain skew semigroup rings having coefficients in A, rather than in R.

The above remarks notwithstanding, we *are* able to obtain, for rather general semigroups, concrete descriptions of rings of the form $(R\#S^*) * S^*$ and $S^* * (R\#S^*)$ in terms of the coefficient ring R. Specifically, in [2] we generalize the group-theoretic results of Albu and Năstăsescu by describing these smash–skew rings as specific types of rings of endomorphisms of a canonical graded module.

The authors are extremely grateful to the referee, whose insightful comments helped them to significantly improve both the mathematical content as well as the presentation of the original version of this manuscript.

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